Reflection of Light from Semi-Infinite Absorbing Turbid Media. Part 2: Plane Albedo and Reflection Function

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Abstract: *We study the accuracy of a number of simple approximate equations for the reflection characteristics of semi-infinite light scattering media (e.g., for the diffuse reflectance of a turbid layer under collimated illumination conditions and also for the reflection function). Spectra of diffuse reflection under monodirectional illumination by a wide beam* $\Re$ (ξ) *or the plane albedo are handled in technological applications using different approximations depending on the level of light absorption in the medium. Generally speaking, the value of* $\Re$ *depends not only on the cosine of incidence angle* ξ *and the similarity parameter* s $= \sqrt{(1-\omega_0)(1-\omega_0 g)^{-1}}$ *(ω_0 is the single scattering albedo,* g *is the asymmetry parameter), but also on the phase function of a turbid layer. The dependence on the phase function is reduced to the dependence on the asymmetry parameter for the low probability of photon absorption in the medium. We find that the influence of the angular behavior of the phase function cannot be neglected in calculations of the reflection function. Then the best choice is the solution of the nonlinear integral equation for the reflection function. We also show a number of approximate results, which can be used to model the reflection function. The results obtained can be used for the solution of both direct and inverse problems of light scattering media optics.*

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INTRODUCTION

The basic idea of numerous applications of the radiative transfer theory[1] for technology and optical engineering is that one must choose the experimental set-up in such a way that simple approximations become valid and can be used effectively for the solution of the inverse problem.[2] This can reduce the numerical burden and increase the accuracy of techniques considerably. Therefore, spectral measurements of the spherical albedo[3] r (e.g., using an integrating sphere) are highly advisable not only for the spectroscopy of turbid media, but also for the optimization of optical properties of a given medium with respect to its microstructure and chemical composition, for design of coloured turbid media and numerous other applications. This is mostly due to the fact that highly accurate approximations and parameterizations are available for the spherical albedo as discussed by Kokhanovsky and Sokoletsky[3] (Part 1).

However, in some cases directional properties of reflected light are of interest. An example of such a case is the use of remote sensing observations and algorithms for the solution of numerous physical and bio-optical problems. The task of this article is to present a number of analytical equations, which can be used to relate the similarity parameter $s = \sqrt{(1-\omega_0)(1-\omega_0 g)^{-1}}$ (ω_0 is the single scattering al-

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bedo defined as the ratio of the scattering coefficient to the extinction coefficient, g is the asymmetry parameter defined as the average cosine of the scattering angle θ) with the diffuse reflectance $\Re$ under the directional illumination conditions. $\Re$ is also called the plane albedo.[4] Due to the reciprocity principle, the same equation is valid for the diffuse illumination and observation from the direction of light beam incidence.

The plane albedo $\Re(\xi)$ describes the relative amount of reflected light for a given incidence angle $\vartheta_0 = arccos(\xi)$. In particular, it is equal to the ratio of energy leaving a scattering layer through unit area in unit time to the incident energy penetrating to the same medium through the same area and time. Clearly, if a semi-infinite medium does not absorb light, then we have: $\Re(\xi) = 1$ by definition. It means that all incident photons must return to outer space. Also, we have for the absolute black surface: $\Re(\xi) = 0$. So photons cannot leave the medium. In practice, $\Re$ takes values between these two limits. In this respect, it is similar to the spherical albedo $r = 2\int_0^1 \Re(\eta)\eta d\eta$, where η is the cosine of the observation angle ϑ. However, the plane albedo is more sensitive to the phase function $p(\theta)$ and cannot be described by just one parameter $s\epsilon[0,1]$ as it is in the case of r. This makes simple and accurate parameterizations such as those described in Part 1[3] difficult to derive. However, a number of useful approximations can be obtained for weak (bright media) and strong (dark media) absorption cases. They are considered in the next section.

The important question related to approximations of the reflection function is discussed in the section "Reflection Function" of this article. The results derived in the latter and Appendix can be used to model bidirectional properties of turbid media.

PLANE ALBEDO

Weak Absorption

We will consider now an important approximation for the plane albedo $\Re$ of weakly absorbing strongly scattering turbid layers (bright surfaces and colours). For this, we present the plane albedo as a Taylor series with respect to the single scattering albedo ω_0. Then it follows:

$$\Re(\omega_0) = \sum_{j=0}^{\infty} A_j \omega_0^j, \tag{1}$$

where we have omitted the dependence of $\Re$ and A_j on ξ for the sake of simplicity. Clearly, it follows by definition:

$$\sum_{j=0}^{\infty} A_j = 1. \tag{2}$$

The representation (1) is helpful for the case of $\omega_0 \rightarrow 0$. Then only a small number of terms is of importance in the general expression (1). For weakly absorbing media, it is necessary to consider the expansions with respect to a small parameter $\beta = 1 - \omega_0$. Therefore, we have instead of Eq. (1):

$$\Re(\omega_0) = \sum_{j=0}^{\infty} A_j (1-\beta)^j, \tag{3}$$

where $(1 - \beta)^j$ can be presented as:

$$(1-\beta)^j = \sum_{i=0}^{j} (-1)^i \frac{j!\beta^i}{i!(j-i)!}. \tag{4}$$

The substitution of Eq. (4) in Eq. (3) gives:

$$\Re(\omega_0) = \sum_{j=0}^{\infty} A_j \left\{ 1 - j\beta + \frac{j(j-1)}{2}\beta^2 - \cdots \right\} \tag{5}$$

or taking into account Eq. (2) :

$$\Re(\omega_0) = 1 - \langle j \rangle \beta + \frac{\langle j(j-1) \rangle}{2}\beta^2 - \cdots, \tag{6}$$

where $\langle n(j) \rangle \equiv \sum_{j=0}^{\infty} n(j) A_j$ for arbitrary $n(j)$. Clearly, the main contribution to $\Re\ (\omega_0)$ comes from the region of large values of j for weakly absorbing media. So we can assume that $\langle j(j-1) \rangle \approx \langle j^2 \rangle$ and similar to higher order terms. This allows us to derive the following exponential approximation for the plane albedo:

$$\Re(\omega_0) = <\ \exp(-\tilde{j}\beta)\ > \tag{7}$$

or

$$\Re(\omega_0) = \exp(-\tilde{j}(\beta)\beta), \tag{8}$$

where we used the mean value theorem and the unknown function $\tilde{j}(\beta)$ must be determined using either exact radiative transfer calculations or the approximate results of the radiative transfer equation valid as $\beta \rightarrow 0$.

In particular, it is known that[4]

$$\Re(\omega_0, \xi) = 1 - \frac{4k(\omega_0)u_0(\xi)}{3(1-g)} \tag{9}$$

as $\beta \rightarrow 0$. Here $g = \frac{1}{2}\int_0^{\pi} p(\theta)\sin\theta\cos\theta\, d\theta$ is the asymmetry parameter, $k(\omega_0)$ is the diffusion exponent,[2] and $u_0(\xi)$ is the escape function[4] for nonabsorbing semi-infinite media. The expansion of Eq. (8) in series and comparison of first two terms of this expansion with Eq. (9) gives:

$$\tilde{j}(\beta) = \frac{4k(\omega_0)u_0(\xi)}{3\beta(1-g)} \tag{10}$$

and, therefore,

$$\Re(\omega_0, \xi) = \exp\left\{ -\frac{4k(\omega_0)u_0(\xi)}{3(1-g)} \right\}. \tag{11}$$

The diffusion exponent k can be presented as[4]

$$k = \sqrt{3(1-\omega_0)(1-g)} \tag{12}$$

for weakly absorbing media with arbitrary phase functions. So we have instead of Eq. (11):

$$\Re(\omega_0,\xi) = \exp\{-\sigma s u_0(\xi)\}, \tag{13}$$

where $\sigma = 4/\sqrt{3}$, $s = \sqrt{(1-\omega_0)/(1-g\omega_0)}$ and we used the fact that $\omega_0 g \approx g$ as $\omega_0 \to 1$. It follows from Eq. (13) that $s = \sigma^{-1}u_0^{-1}(\xi)\ln(\Re^{-1}(\xi))$. So we can relate spectra of s (λ) and $\Re(\lambda)$ for a given ξ and $s \to 0$. This reduces the problem of colour matching from multiple light scattering case to that of single scattering as discussed by Kokhanovsky and Sokoletsky.[3]

Clearly, repeating the same steps but for the spherical albedo r, we obtain:

$$r = \exp\{-\sigma s\}. \tag{14}$$

The exponential approximation (13) reduces the calculation of the plane albedo for weakly absorbing media with a given value of s to the calculation of the escape function $u_0(\xi)$. Equation (13) was proposed by Bushmakova *et al.*[5] However, they used the parameter $\Xi = \sqrt{\frac{1-\omega_0}{1-g}}$ instead of s in Eq. (13). This parameter is approximately equal to s at small values of β.

An important point is that $u_0(\xi)$ does not depend on ω_0 by definition and also its dependence on the phase function is rather weak.[2,6] In particular, one can use the following simple approximation[2]:

$$u_0(\xi) = \frac{3}{7}(1 + 2\xi) \tag{15}$$

valid with the accuracy better than 2% at $\xi > 0.2$. Then we have in the framework of the exponential approximation (EA):

$$\Re(\omega_0,\xi) = \exp\{-b(1 + 2\xi)s\}, \tag{16}$$

where $b = \frac{4\sqrt{3}}{7} \approx 1$. Therefore, one can also use the following simple equation at small values of s:

$$\Re(\xi) = \exp\{-(1 + 2\xi)s\}. \tag{17}$$

The accuracy of Eq. (16) is studied in Figs. 1 and 2. The relative error shown in Fig. 2 is defined as $\varepsilon = 100(\Re_{ap}/\Re_{ex} - 1)$, where $\Re_{ap}(\Re_{ex})$ is the approximate (exact) value of the plane albedo. Exact results were obtained using the solution of the radiative transfer equation as discussed by Mishchenko *et al.*[7] (see Appendix). Note that both figures have been plotted for the five phase functions from Table I in Part 1.[3] Generally, accuracy deteriorates with increasing s. An explanation of this follows from the fact that the exponential approximation was derived in assumption of $\omega_0 \to 1 (s \to 0)$. It follows from the analysis of Figs. 1 and 2 that the exponential approximation is valid with the accuracy better than 22% at $s \leq 0.6$ and $\vartheta_0 \leq 80°$; the error is below 1% at $s \leq 0.2, \vartheta_0 \leq 80°$.

Generally, the error increases with the zenith incidence angle. This can be easily understood. Indeed, for grazing incidence conditions, the contribution of single scattering is larger, and this contribution is not accounted for by the exponential approximation in a correct way. The approximation presented here has better accuracy for large contributions of multiple light scattering in the registered signal.

Equation (16) gives a correct dependence of $\Re$ on ξ signifying that the reflectance is larger for grazing incidence angles (small values of ξ), when photons have smaller chances to be absorbed by the medium in question. This is also confirmed by exact calculations and experimental observations.[8]

Strong Absorption

The quasi-single-scattering approximation (QSSA) for the reflection function[3] $\Re_\infty(\xi,\eta,\varphi)$ ($\xi = \cos\vartheta_0, \eta = \cos\vartheta, \varphi$ is the relative azimuth) of a semi-infinite turbid layer was proposed by Gordon.[9] It is also called the Gordon's approximation. The idea of QSSA is based on the fact that the scattering phase function $p(\theta)$ is highly peaked in the forward ($\theta = 0$) direction in many disperse media, so most part of the scattered light remains in the beam, and the loss is only due to absorption and backscattering. Also it is assumed that the probability of photon absorption β is not small. The correspondent equation for the reflection function is derived in Appendix. It has the following form[9]:

$$R_\infty(\xi,\ \eta,\ \varphi) = \frac{\omega_0 p(\theta)}{4(\xi + \eta)(1 - \omega_0 F)}, \tag{18}$$

where $F = 1 - B$. Here $B = \frac{1}{2}\int_{\pi/2}^{\pi} p(\theta)\sin\theta d\theta$ is the backscattered fraction.[3] The scattering angle θ can be presented as[4]

$$\theta = arc\cos\{-\xi\eta + \sqrt{(1 - \xi^2)(1 - \eta^2)}\cos\varphi\}. \tag{19}$$

We underline that θ is determined simply as $\pi - \vartheta$ under normal illumination.

The plane albedo is defined as[2]

$$\Re(\xi) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 R(\xi,\eta,\varphi)\eta d\eta d\varphi. \tag{20}$$

Therefore, we need to calculate the integral [see Eqs. (18), (20)]:

$$\Phi = 2\int_0^1 \bar{p}(\xi, -\eta)(\xi + \eta)^{-1}\eta d\eta, \tag{21}$$

where $\bar{p}$ is azimuthally averaged phase function. It is known that $\bar{p}$ can be expressed via the following series[2]:

$$\bar{p}(\mu_0,\mu) = \sum_{j=0}^{\infty} x_j P_j(\mu_0)P_j(\mu), \tag{22}$$

where $P_j(\mu)$ are the Legendre polynomials and x_j are coefficients of the expansion of the phase function as specified below:

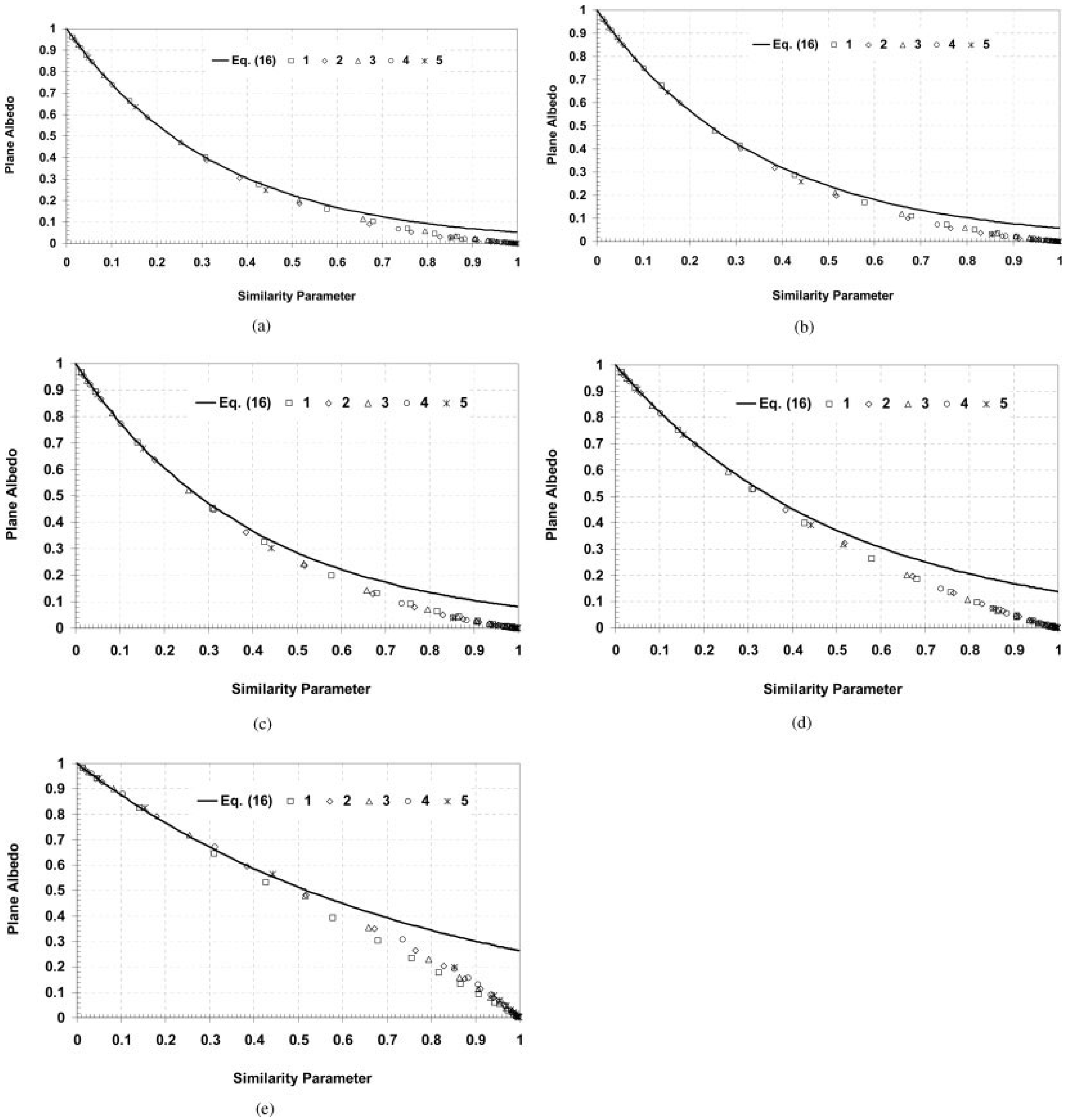


FIG. 1. Dependence of the plane albedo on the similarity parameter for the five phase functions shown in the legend and listed in Table I of Ref. 3 at the illumination angle 0° (a), 20° (b), 40° (c), 60° (d), and 80° (e). All calculations have been carried out using the exact radiative transfer theory (symbols) and Eq. (16) (solid curve). The values of the asymmetry parameter g for selected five phase functions were equal to 0.5033 (1), 0.6962 (2), 0.8546 (3), 0.9062 (4), 0.9583 (5).

$$p(\theta) = \sum_{j=0}^{\infty} x_j P_j(\theta). \tag{23}$$

The substitution of Eq. (22) in Eq. (21) gives:

$$\Phi = 2\sum_{j=0}^{\infty} (-1)^j x_j P_j(\xi) Q_j(\xi), \tag{24}$$

where

$$Q_j(\xi) = \int_0^1 P_j(\eta)(\xi + \eta)^{-1}\eta d\eta \tag{25}$$

and we used the property:

$$P_j(-\eta) = (-1)^j P_j(\eta). \tag{26}$$

Therefore, one can obtain using Eqs. (18) and (20):

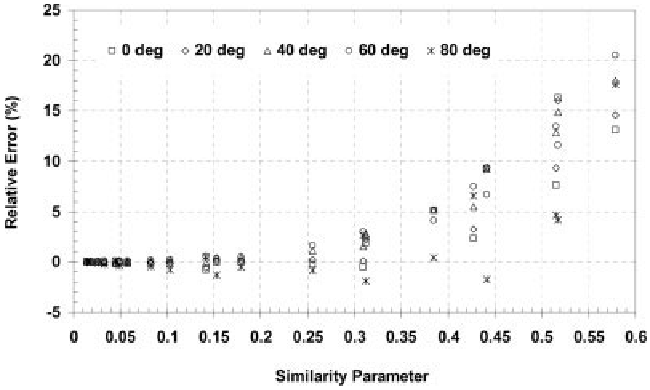


FIG. 2. Dependence of the relative error of the exponential approximation, Eq. (16), on the similarity parameter $s \leq 0.6$ as compared with exact radiative transfer calculations for different illumination angles (in degrees) and selected phase functions.

$$\Re(\xi) = \frac{\omega_0}{2(1-\omega_0 F)} \sum_{j=0}^{\infty} (-1)^j x_j P_j(\xi) Q_j(\xi). \qquad (27)$$

This equation allows to substitute the solution of the radiative transfer equation (see Appendix) by a simple summation of series. The integration of this equation with respect to ξ allows us to obtain an accurate approximation for the spherical albedo valid as $s \to 1$ (strong absorption). Namely, it follows: $r = \Im \omega_0 (1-\omega_0 F)^{-1}$, where the value of $\Im = \sum_{j=0}^{\infty} (-1)^j b_j x_j$ and $b_j = \int_0^1 P_j(\xi) Q_j(\xi) \xi d\xi$.

We study the accuracy of Eq. (27) as compared with the exact results in Figs. 3(a)–(c) and Fig. 4 for three selected phase functions. The relative error as defined above generally decreases with the increase in the similarity parameter s: this means decreasing ω_0 or increasing g or both.

The impact of ω_0 on the plane albedo is much stronger than the impact of g for the cases studied. In particular, at any phase function and $\omega_0 \leq 0.3$ (i.e., $0.91 \leq s \leq 0.99$ for selected phase functions), it follows that $\varepsilon \leq 13\%$ at $\vartheta_0 \leq 70°$. One obtains that $\varepsilon \leq 35\%$ at $\vartheta_0 \leq 70°$ and $\omega_0 \leq 0.7$ (i.e., $0.68 \leq s \leq 0.95$ for selected phase functions).

The errors of Eq. (27) increase for small and intermediate values of s. This is related to the fact that Eq. (18) is valid for strongly absorbing media only ($s \to 1$). Equation (16) can be used at small values of s. The next section is devoted the derivation of approximate equations valid at arbitrary ω_0.

Arbitrary Absorption

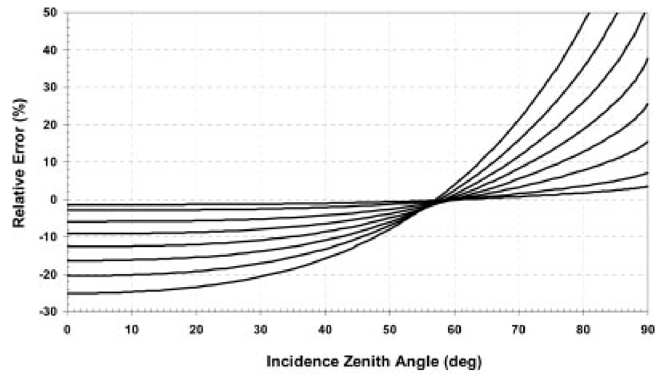
Let us consider now the plane albedo for semi-infinite turbid plane–parallel layers at arbitrary ω_0. We start from the approximate expression[10] valid for the isotropic scattering case ($p(\theta) \equiv 1$, and therefore, $g = 0$):

$$\Re(\xi) = 1 - \sqrt{1-\omega_0} H(\xi), \qquad (28)$$

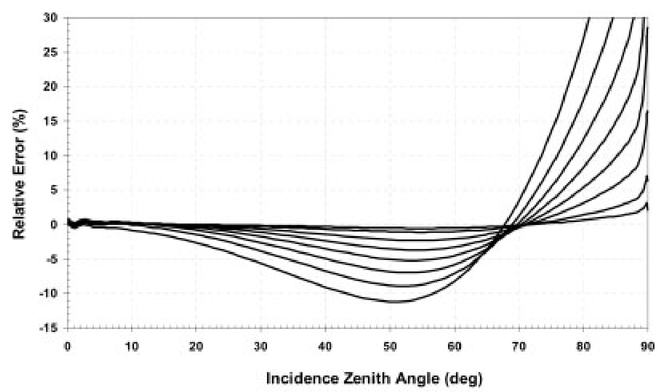
where the Ambartsumian–Chandrasekhar function $H(\xi)$ is often approximated as[10,11]

$$H(\xi) = \frac{1+2\xi}{1+2\xi\sqrt{1-\omega_0}}. \qquad (29)$$

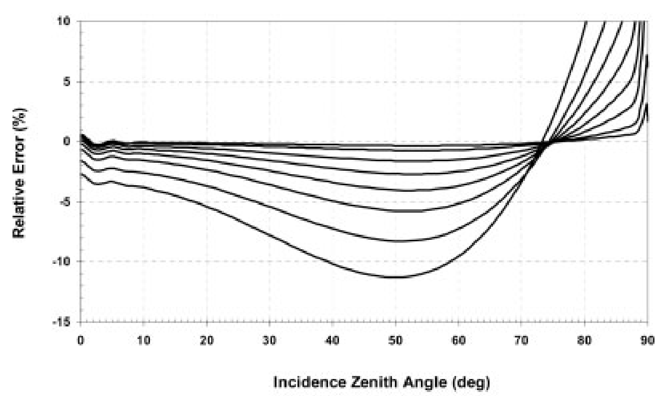
Eqs. (17) and (28) with account for Eq. (29) have a common limit as $\omega_0 \to 1$, $g \to 0$. To generalize Eqs. (28) and (29) for media with arbitrary phase functions, we apply the Hulst's similarity rule[12,13] using the replacement the single-scatter-



(a)



(b)



(c)

FIG. 3. Dependence of the relative error of Eq. (27) on the illumination angle for the phase functions with $g = 0.5033$ (a), $g = 0.8546$ (b), and $g = 0.9583$ (c) (see Table I in Ref. 3) calculated for different values of $\omega_0 = 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7$. The curves with smaller values of $|\varepsilon|$ at $\vartheta_0 = 0°$ correspond to smaller values of ω_0.

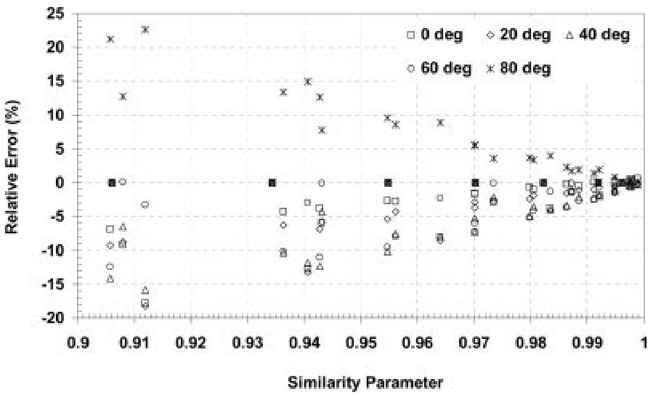


FIG. 4. Dependence of the relative error of Eq. (27) on the similarity parameter s.

ing albedo ω_0 by the reduced single scattering albedo ω_0' as follows:

$$\omega_0' = \frac{\sigma_{\text{sca}}(1-g)}{\sigma_{\text{abs}} + \sigma_{\text{sca}}(1-g)} = \frac{1-g}{\omega_0^{-1} - g} \equiv 1 - s^2, \tag{30}$$

where σ_{abs} is the absorption coefficient and σ_{sca} is the scattering coefficient. Therefore, we obtain:

$$\Re(\xi) = \frac{1-s}{1+2\xi s}. \tag{31}$$

Equation (31) coincides with the correspondent formula derived by Hapke[10] for the case of isotropic scattering. It follows from Eq. (31) that $\Re \to 0$ as $s \to 1$. This is a correct limit for absolutely black surfaces. Interestingly, Eqs. (31) and (17) give the same asymptotics for weakly absorbing media $s \to 0$:

$$\Re(\xi) = 1 - (1 + 2\xi)s. \tag{32}$$

However, Eq. (17) does not provide a correct limit as $s \to 1$.

Equation (31) is very accurate for a turbid layer illuminated by collimated light under angle equal to 60°. Then the plane albedo is the function (almost solely) of the similarity parameter s, see Fig. 5. In this case ($\xi = 0.5$) root-mean-squares errors are smaller than 2% for the overall comparison dataset. Therefore, the illumination of a layer at 60° with respect to the normal is required for an accurate interpretation of measurements using Eq. (31). It follows from Eq. (31) at $\xi = 0.5$ that $s(\lambda) = (1 - \Re(\lambda)) \times (1 + \Re(\lambda))^{-1}$. This enables a reduction of a general problem from multiple to the single scattering case.[3]

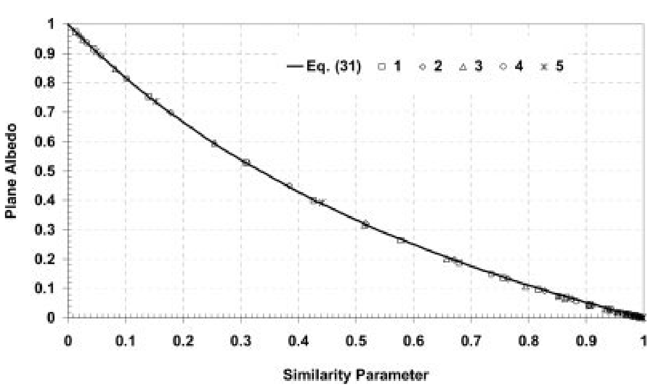
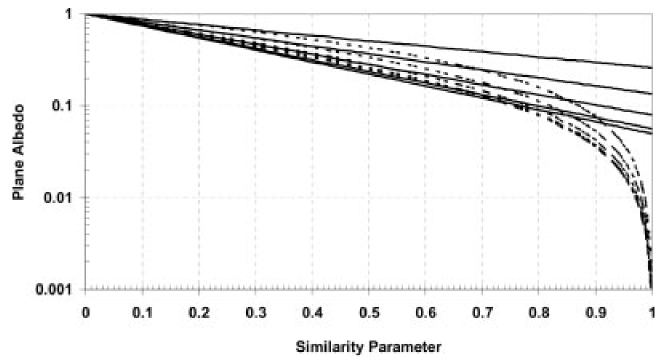
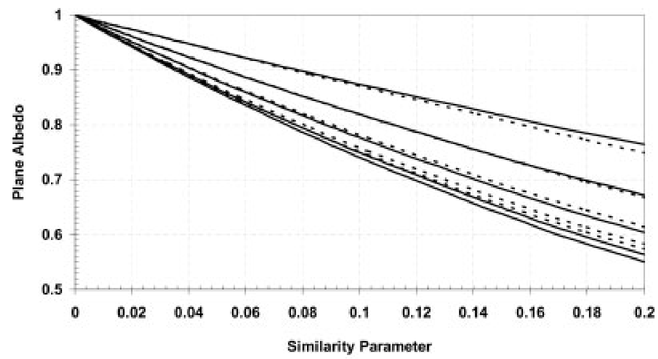


FIG. 5. Dependence of the plane albedo on the similarity parameter at the illumination angle 60° (solid curve: Eq. (31), symbols: exact calculations for different phase functions as in Fig. 1).



(a)



(b)

FIG. 6. Dependence of the plane albedo on the similarity parameter for several illumination angles calculated using Eq. (17) (solid lines) and Eq. (31) (dotted lines). All of the plots have been obtained for illumination angles 0°, 20°, 40°, 60°, and 80° (from the bottom to the top), and for similarity parameters <1 (a) or <0.2 (b).

We compare results of numerical calculations using Eqs. (17) and (31) in Fig. 6(a). It follows that Eq. (31) describes the general behavior of $\Re$ with respect to s in a correct way (e.g., $\Re = 0$ at $s = 1$). The approximation (17) is valid only for weakly absorbing media. The same comparison but for $s < 0.2$, where Eq. (17) gives accurate results, is shown in Fig. 6(b). It follows that Eq. (31) gives similar results as Eq. (17) at $s < 0.2$ for the illumination zenith angle equal to 60°.

Figure 7 yields a comparison of errors for Eqs. (17) and (31) at two important angles, 45° (an angle popularly used in colour matching practice) and 60° (an angle at which plane albedos are very close to the spherical albedos and also popular in colour matching). We see that both equations can be used in these cases with errors smaller than 5% at $s < 0.4$.

We can find the analytical equation for the spherical albedo $r = 2\int_0^1 \Re(\xi)\xi d\xi$ using Eq. (31). This gives

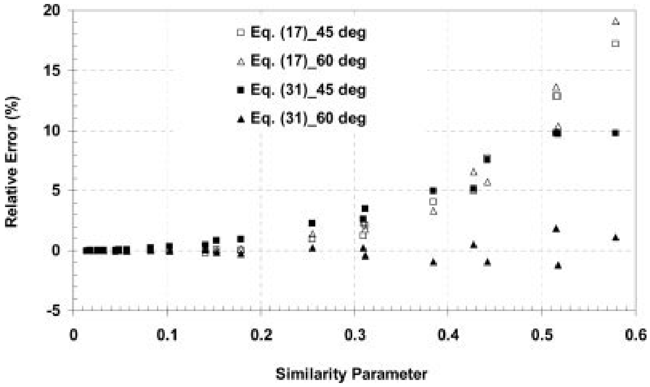


FIG. 7. Relative errors of Eqs. (17) and (31) as compared with exact radiative transfer calculations at the illumination angles 45° and 60°, and five selected phase functions.

$$r = \frac{1-s}{s}\left(1 - \frac{\ln(1+2s)}{2s}\right). \qquad (33)$$

We present the relative error of the spherical albedo as derived using this equation in Fig. 8 (for 5 phase functions with different asymmetry parameters). In the same figure, we show the difference (in percent) between Eq. (33) and the van de Hulst approximation (HA)[12,13] [$r = (1 - 0.139s)(1 - s)/(1 + 1.17s)$]. The errors are largest for dark ($s \to 1$) scattering media. The accuracy of the van de Hulst approximation is lower in this case (see Ref. 3, Fig. 3). The error of Eq. (33) is only weakly sensitive to the phase function of a scattering medium. The value of the spherical albedo as calculated using Eq. (33) gives accurate results (close to results yielded by HA) in the case of $s < 0.8$. Errors of Eq. (33) are larger than those of HA at $s \geq 0.8$. Interestingly, the results obtained using both approximations are almost the same for the phase function with $g = 0.8546$ (phase function 3 from the Table I of Ref. 3) at any s.

As $s \to 0$, we have from Eq. (33)

$$r = 1 - \sigma s \qquad (34)$$

where $\sigma = 7/3$. This differs from a correct asymptotic result given by Eq. (34) with $\sigma = 4/\sqrt{3}$ (see Ref. 4) by less than 1% at $s < 0.21$. Note that Eq. (34) enables us to make the physical interpretation of the similarity parameter s as $s \to 0$. Namely, σs gives the absorptance of the medium $A_{abs} = 1 - r$ for the diffuse illumination and observation conditions. This explains the reason why the similarity parameter increases with the probability of photon absorption β. It increases with g because photons can penetrate to larger depths for highly extended in forward direction phase functions ($g \to 1$). This makes it difficult for photons to escape from the absorbing medium. Therefore, the design of dark media requires not only highly absorptive additives but also particulates with large asymmetry parameters having forward-extended phase functions.

It follows from Figs. 5 and 7 that the accuracy of Eq. (31) is the best at $\vartheta_0 = 60°$. This suggests that the accuracy at arbitrary values of ϑ_0 can be increased using the multiplicative factor $\Phi(\zeta)$, where $\zeta = \xi - 0.5$, with the property $\Phi(0) = 1$. Namely, it follows:

$$\Re(\xi) = \Phi(\zeta)\frac{1-s}{1+2s\xi}, \qquad (35)$$

where we have fitted exact radiative transfer calculations for phase functions shown in Table I of Ref. 3 to obtain:

$$\Phi(\zeta) = \exp\{[A_1\zeta + A_2\zeta^2]s + [A_3\zeta + A_4\zeta^2]s^2\}, \qquad (36)$$

where $A_i = \sum_{j=1}^{3} \alpha_{ij} g^{j-1}$ and the elements α_{ij} can be deduced from the following matrix:

$$\hat{\alpha} = \begin{pmatrix} -0.991 & 3.139 & -1.874 \\ 1.435 & -4.294 & 2.089 \\ 0.719 & -5.801 & 2.117 \\ -0.509 & 0.418 & 3.360 \end{pmatrix}. \qquad (37)$$

The accuracy of Eq. (35) (we shall call this approximation "an extended Hapke approximation," see Eq. (31) as well) is shown in Fig. 9. It follows that the maximal error of Eq. (35) is smaller than 10% at $s \leq 0.75$ [Fig. 9(a)]. It is important that Eqs. (35)–(37) especially suitable for the ordinary conditions of colour matching, namely, illumination angles $40° \leq \vartheta_0 \leq 60°$ and asymmetry parameters $0.45 \leq g \leq 0.55$ [see Fig. 9(b)] for the phase function with $g = 0.5033$). The relative error increases for larger values of s (partially, because $\Re(\xi) \to 0$ as $s \to 1$ At $s > 0.9$, Eq. (27) is preferable for any phase function [compare Figs. 9(a) and 9(b) with Fig. 4].

We conclude that Eq. (35) can be used for rapid estimations of the plane albedo variation for arbitrary parameters of light scattering media. The weak absorption approximation (see "Weak absorption" under "Plane albedo") is preferable as $s \to 0$ and Eq. (27) is valid as $s \to 1$. The exact radiative transfer calculations must be used if still higher accuracy is desired.

For a number of applications, it is important to derive the illumination angle ϑ_0 at which the plane albedo coincides with the spherical albedo. We found using exact radiative transfer calculations that ϑ_0 is in the range of 48–61°

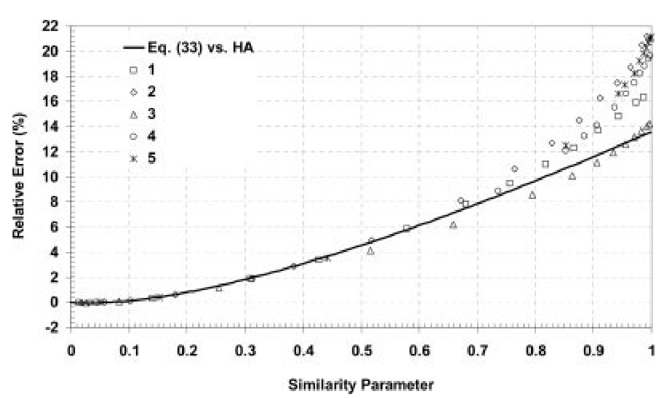


FIG. 8. Relative errors of Eq. (33) as compared with the exact calculations for different phase functions (symbols as in Fig. 1) and to the van de Hulst approximation for the spherical albedo (solid curve).

TABLE I. Coefficients of the equation $\vartheta_0(s) = a+bs+cs^2+ds^3$. The coefficient of determination R^2 (showing the quality of the fit) and the asymmetry parameter g are given as well.

No.	g	a	b	c	d	R^2
1	0.5033	48	10.31	−9.903	6.078	0.964
2	0.6962	48	11.80	−20.28	17.61	0.984
3	0.8546	48	13.13	−15.77	14.41	0.989
4	0.9062	48	20.30	−40.31	32.86	0.993
5	0.9583	48	15.04	−27.59	25.23	0.992

Here ϑ_0 (in degrees) is the root of equation $\Re(\vartheta_0) = r$ for the five different phase functions (see Table 1 of Ref. 3). A fixed value of $a = 48°$ valid for any $s > 0$ was used for the root-mean-squares fitting. Note that in the case of conservative scattering ($s = 0$), $\Re(\vartheta_0) = r = 1$ at any angle ϑ_0.

depending on $s \neq 0$ and the phase function $p(\theta)$. In particular, it follows that $\vartheta_0(s)$ may be approximated by the cubic polynomial $\vartheta_0(s) = a + bs + cs^2 + ds^3$, where coefficients a, b, c, d along with the coefficient of determination usually used in the regression analysis are listed in the Table I for different phase functions and $s \neq 1$. Clearly, it follows that $r \equiv \Re = 1$ at $s = 0$ for an arbitrary incidence angle. This is due to the conservation of energy law.

REFLECTION FUNCTION

Weak Absorption

Reflection function (RF) is defined as the ratio of the intensity of light reflected from a given turbid medium to that for an absolutely white Lambertian surface. This function can be calculated using the nonlinear integral radiative transfer equation introduced by Ambartsumian[14] (see Appendix). The correspondent code is available online.[7] Basically, it is much more difficult to model RF as compared with plane or spherical albedos. The dependence of RF on the phase function cannot be ignored. Therefore, numerical calculations must be used in this case. A number of approximate relations for the reflection function of a semi-infinite layer is given in Table II.

In particular, following the same steps as in derivation of Eq. (11) for the plane albedo, we obtain:

$$R_\infty(\xi,\eta,\varphi) = R_{0\infty}(\xi,\eta,\varphi)\ \exp\left\{-\frac{4s}{\sqrt{3}}R_{0\infty}^{-1}(\xi,\eta,\varphi)u_0(\xi)u_0(\eta)\right\} \tag{38}$$

Here $R_{0\infty}(\xi,\eta,\varphi)$ is the reflection function of a semi-infinite nonabsorbing medium[15] (see Table II) having the same phase function as an absorbing layer under study. This equation was first proposed by Rozenberg,[16] but he did not give the relation of the parameter s to the local optical characteristics of a turbid layer. Equation (38) provides accurate estimations[2,18] of the reflection function as $s \to 0$. Also it follows from Eq. (38):

$$s(\lambda) = \frac{\sqrt{3}R_{0\infty}(\xi,\eta,\varphi)}{4u_0(\xi)u_0(\eta)}\ln\left\{\frac{R_{0\infty}(\xi,\eta,\varphi)}{R_\infty(\xi,\eta,\varphi)}\right\} \tag{39}$$

which can be used in colour matching applications.

The accuracy of Eq. (38) is studied in Fig. 10. We see that the accuracy is better than 10% in most of cases at $\omega_0 > 0.99$. We also found that the accuracy does not depend substantially on the value of g. However, it is strongly influenced by ω_0. Errors can reach 20% at $\omega_0 = 0.95$.

Strong Absorption

The Gordon's approximation[9] (see Eq. (18) and Appendix) is well suited for calculations of RF in the case of strong absorption. The accuracy of this approximation (see Eq. (18)) for two extended in the forward direction phase functions with $g = 0.9062$ and $g = 0.9583$ under normal viewing[3] is shown in Figs. 11 and 12 as the function of the incidence zenith angle. We conclude that the error of the Gordon's approximation decreases with $\beta = 1 - \omega_0$. The error is below 13% for the case studied in Fig. 11(b) ($g = 0.9583$) at $\omega_0 \leq 0.75$ for almost all incidence angles (except the cases $\vartheta \to \vartheta_0$ where glint and backscattering enhancement effects may play a role and also at $\vartheta \to \pi/2$). The decrease in the error for smaller values of ω_0 is due to

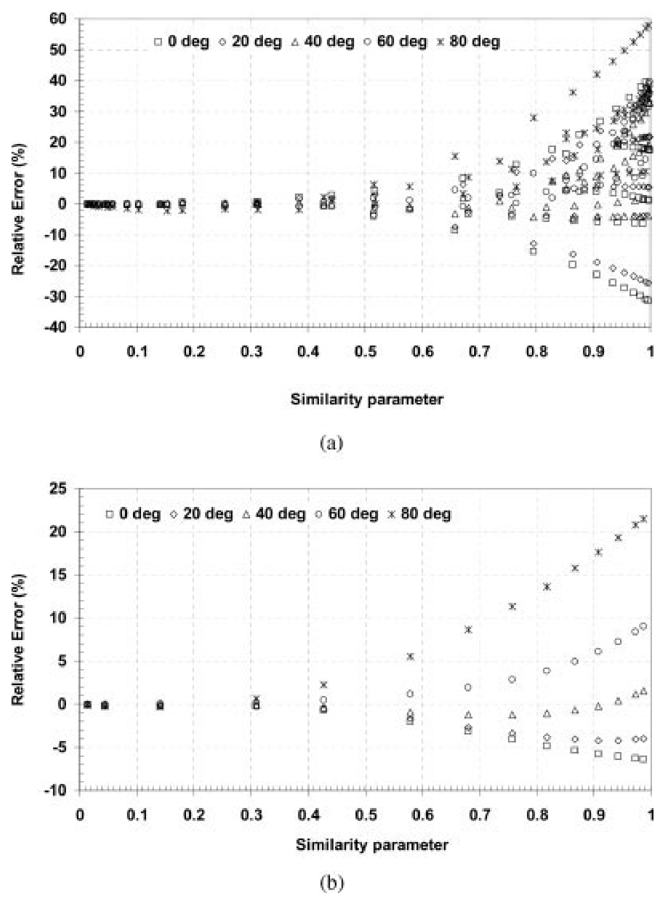


FIG. 9. Relative errors of an extended Hapke approximation, Eq. (35), as compared with exact radiative transfer calculations for different illumination angles (symbols as in Fig. 2) and for (a) all phase functions; (b) the phase function with $g = 0.5033$.

TABLE II. Approximations for the reflection function of a semi-infinite turbid layer.

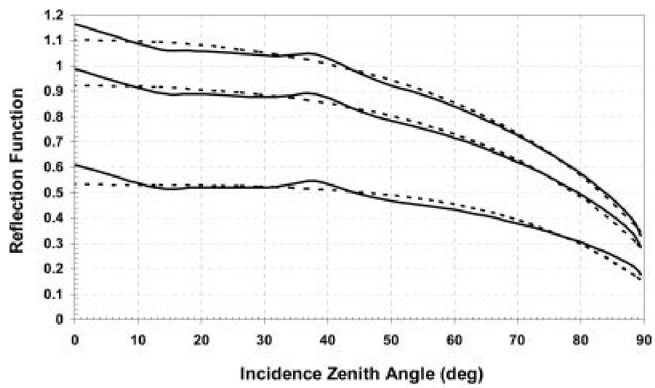
Single scattering albedo	Reflection function	Parameters and remarks	References
$\omega_0 = 1$	$R_{0\infty}(\xi,\eta,\varphi) = \dfrac{A + B(\xi+\eta) + C\xi\eta}{4(\xi+\eta)} + \dfrac{H(\theta)}{4(\xi+\eta)}$, $H(\theta) = p(\theta) - \bar{p}(\theta)$, $\bar{p}(\theta) = \sum_{j=1}^{\infty} (-1)^j x_j P_j(\xi) P_j(\eta)$	A=3.944, B=−2.5, C = 10.664 (for water clouds)	15
$\omega_0 \to 1$	$R_{\infty}(\xi,\eta,\varphi) = R_{0\infty}(\xi,\eta,\varphi) \times \exp\left[-\dfrac{4s}{\sqrt{3}} R_{0\infty}^{-1}(\xi,\eta,\varphi) u_0(\xi) u_0(\eta)\right]$	$R_{0\infty}(\xi,\eta,\varphi)$ can be taken from the line above (for water clouds) or from the work of Melnikova et al.[16] (for arbitrary g)	5, 6, 16, 18
$\omega_0 \to 0$	$R_{\infty}(\xi,\eta,\varphi) = \dfrac{\omega_0 p(\theta)}{4(\xi+\eta)}$, $\theta = \arccos[-\xi\eta + \sqrt{(1-\xi^2)(1-\eta^2)}\cos\varphi]$		1
$\omega_0 \leq 0.7$	$R_{\infty}(\xi,\eta,\varphi) = \dfrac{\omega_0 p(\theta)}{4(\xi+\eta)(1-\omega_0 F)}$	$g \to 1$	9

the fact that the Gordon's equation is better suitable for weakly scattering media. In particular, Eq. (18) gives an exact result for single scattering[2] as $\omega_0 \to 0$:

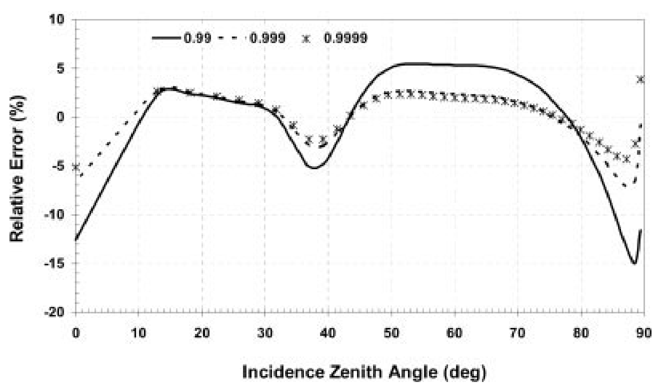
$$R_{\infty}(\xi,\eta,\phi) = \frac{\omega_0 p(\theta)}{4(\xi+\eta)}. \tag{40}$$

We find that Eq. (40) holds with the accuracy better than 8% for $\omega_0 = 0.05$ for all phase functions studied and for all illumination conditions for the normal viewing (with the average error around 5%). The error of Eq. (40) increases with ω_0. In particular, it approximately doubles for $\omega_0 = 0.1$ as compared with the case $\omega_0 = 0.05$. In all cases Eq. (40) gives too low reflectance due to ignoring of double and higher order scattering in the medium. Equation (40) allows us to obtain accurate results for the plane albedo and spherical albedo as $\omega_0 \to 0$. For this one needs to perform correspondent angular integrations as described by Kokhanovsky and Sokoletsky.[3]

Zege *et al.*[6] underline that the error of Eq. (18) is highly dependent on the phase function. It works better for the flat phase functions in the backward scattering hemisphere. Further studies of the accuracy of Gordon's approximation for various phase functions and values of ω_0 are given by Gordon,[9] Golubitsky *et al.*,[19] and Golubitsky and Levin.[20]



(a)



(b)

FIG. 10. (a) Reflection function calculated using Eq. (38) (dotted lines) at $g = 0.8546$ and $\omega_0 = 0.99$, 0.999, 0.9999 (from the bottom to the top). The reflection function for a semi-infinite nonabsorbing cloud was taken from Table II at *H = 0*. Solid curves are obtained using exact radiative transfer calculations. The increased reflection at incidence angles 0 and 40° obtained using exact calculations is due to glory and rainbow[2] scattering effects, respectively. (b) Relative errors correspondent to data shown in Fig. 10(a).

CONCLUSIONS

The plane albedo is only weakly sensitive to the phase function at a given value of the similarity parameter s in the limit $\omega_0 \to 1$. For larger absorption (e.g., for the range $0.7 < \omega_0 < 1$), it is not possible to use just one parameter (e.g., s) to characterize the plane albedo $\Re(\xi)$. Generally, two parameters, e.g., s and g [see Eqs. (35)–(37)] and even the whole set of $\{x_j\}$ [see Eq. (27)] are needed for the accurate

characterization of the angular dependence of the plane albedo (or the diffuse reflectance under directional illumination conditions).

We established approximate limits of the applicability (in terms of the Hulst's similarity parameter s) for the approximate theoretical models considered in the article. Accurate results for the plane albedo are ensured by the exponential approximation, Eq. (16), by the extended Hapke's approximation, Eq. (35), and by the quasi-single-scattering approximation, Eq. (27), at $s \leq 0.6$; $s \leq 0.9$; and $s > 0.9$, respectively.

The reflection function of a semi-infinite turbid layer is determined by the phase function, the single scattering albedo, and three angles (the illumination and observation zenith angles, and the relative azimuth). Therefore, it is much more difficult to parameterize this function. Some results are given in Table II. Generally, the exact radiative transfer equation must be used whenever it is possible to calculate the bidirectional reflectance of a semi-infinite turbid layer.

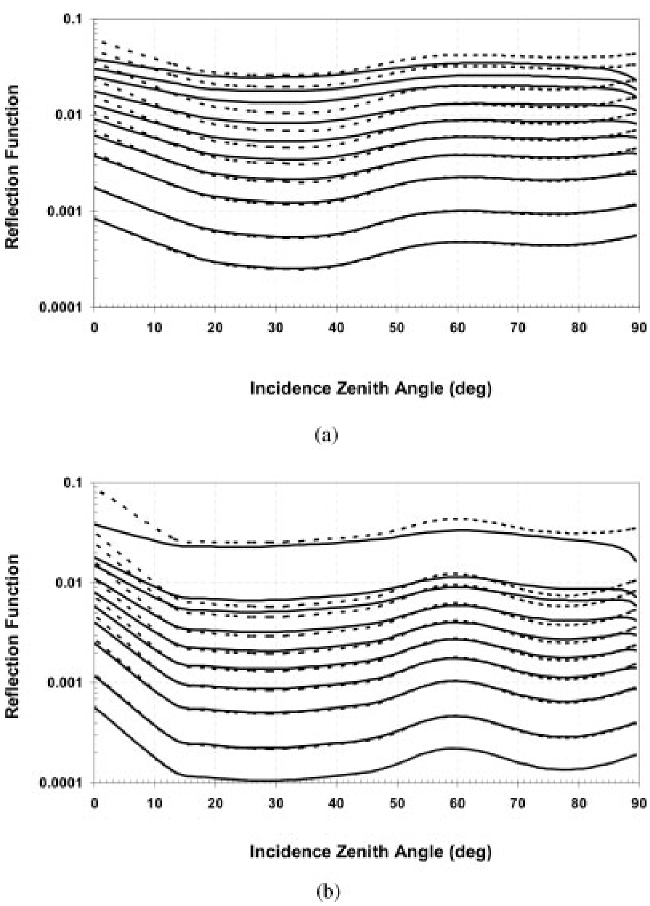


FIG. 11. (a) Reflection function calculated using Eq. (18) (dotted curves) as compared with exact radiative transfer calculations (solid curves) under normal viewing and varied single-scattering albedos (0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.75, 0.8 from below upwards) at $g = 0.9062$ (see Table I in Ref. 3). (b) Reflection function calculated using Eq. (18) (dotted curves) as compared with exact radiative transfer calculations (solid curves) under normal viewing and varied single-scattering albedos (0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.75, 0.9 (from below upwards)) at $g = 0.9583$ (see Table I in Ref. 3).

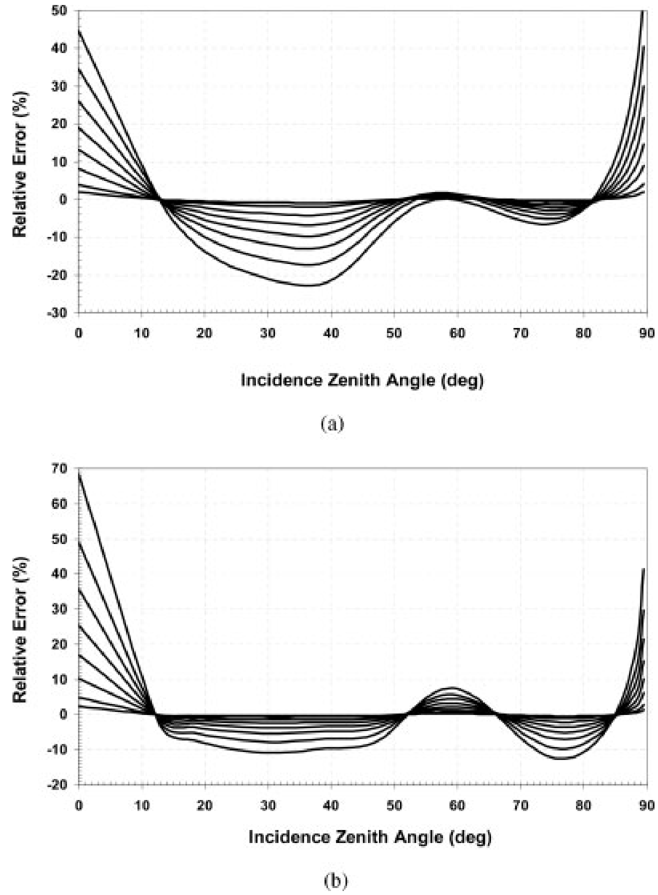


FIG. 12. (a) Relative error of Eq. (18) as compared with exact radiative transfer calculations under normal viewing and varied single-scattering albedos (0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7) at $g = 0.9062$. Smaller absolute errors correspond to smaller values of ω_0 at the illumination zenith angle 30°. (b) Relative error of Eq. (18) as compared with exact radiative transfer calculations under normal viewing and varied single-scattering albedos (0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7) at $g = 0.9583$. Smaller absolute errors correspond to smaller values of ω_0 at the illumination zenith angle 30°.

Equations presented here can be used for colour matching applications and also for design of coloured turbid media having prescribed spectral and angular backscattering characteristics.

APPENDIX: REFLECTION FUNCTION OF STRONGLY ABSORBING MEDIA WITH HIGHLY ANISOTROPIC PHASE FUNCTIONS

This Appendix is devoted to the derivation of the Gordon's approximation for the reflection function $R_\infty(\xi, \eta, \phi, \phi_0)$ of a strongly absorbing semi-infinite medium. We start from the general equation for the reflection function of a turbid plane–parallel homogeneous semi-infinite layer with an arbitrary single scattering albedo ω_0 and the phase function $p(\theta)$. This equation has the following form[14]:

$$R_\infty(\xi,\eta,\phi,\phi_0) = \frac{\omega_0 p(\theta)}{4(\xi+\eta)} + \frac{\omega_0[\xi V(\xi,\eta,\phi,\phi_0) + \eta V(\eta,\xi,\phi,\phi_0]}{4\pi(\xi+\eta)} + \frac{\omega_0 \xi\eta W(\xi,\eta,\phi,\phi_0)}{4\pi^2(\xi+\eta)} \quad \text{(A1)}$$

where

$$V(\xi,\eta,\phi,\phi_0) = \int_0^{2\pi} d\phi' \int_0^1 p(\eta,\phi,\eta',\phi') R_\infty(\xi,\eta',\phi',\phi_0) d\eta' \quad \text{(A2)}$$

$$W(\xi,\eta,\phi,\phi_0) = \int_0^{2\pi} d\phi' \int_0^{2\pi} d\phi'' \int_0^1 d\eta' \int_0^1 R_\infty(\eta,\phi,\eta',\phi') p(-\eta',\phi',\eta'',\phi'') R_\infty(\xi,\eta'',\phi'',\phi_0) d\eta'' \quad \text{(A3)}$$

The numerical solution[7] of this equation was used in this article to check the accuracy of various approximations.

It follows that the reflection function can be presented as

$$R_\infty(\xi,\eta,\phi,\phi_0) = \frac{\omega_0 p(\theta)}{4(\xi+\eta)} + \Pi, \quad \text{(A4)}$$

where the first term accounts for single scattering and the second one gives the contribution of multiple light scattering:

$$\Pi = \frac{\omega_0[\xi V(\xi,\eta,\phi,\phi_0) + \eta V(\eta,\xi,\phi,\phi_0)]}{4\pi(\xi+\eta)} + \frac{\omega_0 \xi\eta W(\xi,\eta,\phi,\phi_0)}{4\pi^2(\xi+\eta)}. \quad \text{(A5)}$$

The integral $V(\xi,\eta,\phi,\phi_0)$ can be simplified for media with highly anisotropic scattering. Then phase functions are highly extended in the forward direction. This means that we assume that $p(\eta,\phi,\eta',\phi')$ is close to the delta function. Then it follows using the definition of the delta function:

$$V(\xi,\eta,\phi,\phi_0) = \int_0^{2\pi} d\phi' \int_0^1 p(\eta,\phi,\eta',\phi') R_\infty(\xi,\eta',\phi',\phi_0) d\eta' \quad \text{(A6)}$$

$$\approx R_\infty(\xi,\eta,\phi,\phi_0) \int_0^{2\pi} d\phi' \int_0^1 p(\eta,\phi,\eta',\phi') d\eta' = 4\pi R_\infty(\xi,\eta,\phi,\phi_0) c(\eta,\phi_0)$$

where

$$c(\eta,\phi) = \frac{1}{4\pi} \int_0^{2\pi} d\phi' \int_0^1 p(\eta,\phi,\eta',\phi') d\eta' \quad \text{(A7)}$$

Now we use the fact that the phase function depends only on the difference $\varphi = \phi - \phi_0$. This means that we have instead of Eq. (A7):

$$c(\eta) = \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_0^1 p(\eta,\eta',\varphi) d\eta'. \quad \text{(A8)}$$

The substitution of Eq. (A6) into Eq. (A5) enables to derive:

$$R_\infty(\xi,\eta,\varphi) = \frac{\omega_0 p(\theta)}{4(\xi+\eta)(1-\omega_0 F^*)}, \quad \text{(A9)}$$

where

$$F^* = \frac{\xi c(\eta) + \eta c(\xi)}{\xi+\eta}. \quad \text{(A10)}$$

The second term in Eq. (A5) was neglected. This is possible because W in Eq. (A5) depends on the product of two reflection functions. The values of R_∞ are small for strongly absorbing media.

Equation (A9) was obtained by Anikonov and Ermolaev.[21] They also have shown that Eq. (A8) can be reduced to the following simpler form:

$$c(\eta) = F - \Delta(\eta), \quad \text{(A11)}$$

where

$$F = \frac{1}{2} \int_0^1 p(\eta') d\eta' \quad \text{(A12)}$$

$$\Delta(\eta) = \frac{1}{2\pi} \int_0^{\sqrt{1-\xi^2}} d\eta \{p(\eta) - p(-\eta')\} arccos \left\{ \frac{\xi\eta}{\sqrt{(1-\xi^2)(1-\eta^2)}} \right\}. \quad \text{(A13)}$$

Our numerical calculations show that $\Delta(\xi) << F$. Therefore, we can neglect the contribution of Δ in Eq. (A11). This reduces Eq. (A9) to the simple formula known as the Gordon's approximation:

$$R_\infty(\xi,\eta,\varphi) = \frac{\omega_0 p(\theta)}{4(\xi+\eta)(1-\omega_0 F)}. \quad \text{(A14)}$$

The accuracy of this equation was studied by Zege,[22] who

also modified Eq. (A14) to account for more complex cases (e.g., the presence of rainbows and glories).

It follows from the derivations presented here that the accuracy of Eq. (A14) increases for highly extended phase functions. But then it follows that $F \rightarrow 1$ and we can write:

$$R_\infty(\xi,\eta,\varphi) = \frac{\omega_0 p(\theta)}{4(\xi + \eta)(1 - \omega_0)}. \quad \text{(A15)}$$

This shows that the angular behavior of the reflection function in the approximation under study is the same as in the single scattering approximation [see Eq. (A4) at $\Pi = 0$]. However, its magnitude is increased by the factor $K = (1 - \omega_0)^{-1}$. This is due to multiple light scattering effects.

It follows that $\omega_0 = \sigma_{sca}/\sigma_{ext}$ and $p(\theta) = \Sigma_{sca}(\theta)/\sigma_{sca}$, where σ_{sca} is the scattering coefficient, $\sigma_{ext} = \sigma_{sca} + \sigma_{abs}$ is the extinction coefficient, σ_{abs} is the absorption coefficient, and $\Sigma_{sca}(\theta)$ is the volumetric angular scattering coefficient. Combining this with Eq. (A15), we obtain $R_\infty(\xi,\eta,\varphi) \propto \Sigma_{sca}(\theta)/\sigma_{abs}$. Thus, one can determine the absorption coefficient as the function of the wavelength λ from measurements of the spectral reflection function. This is especially an easy task for media with wavelength-independent single scattering laws described by the function $\Sigma_{sca}(\theta)$. The formulation given above enables studies of the colour changes (in reflected light) using various additives with different spectra $\sigma_{abs}(\lambda)$.

ACKNOWLEDGMENTS

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1. Chandrasekhar S. Radiative Transfer. New York: Dover; 1960.
2. Kokhanovsky AA. Light Scattering Media Optics. Berlin: Springer-Praxis; 2004.
3. Kokhanovsky AA, Sokoletsky L. Reflection of light from semi-infinite absorbing turbid media. I. Spherical albedo. Color Res Appl 2006;31: 491–497.
4. Sobolev VV. Light Scattering in Planetary Atmospheres. Oxford: Pergamon; 1975.
5. Bushmakova OV, Zege EP, Katsev IL. On asymptotic equations for brightness coefficients of optically thick light scattering layers. Doklady Acad Sci BSSR 1971;15:309–311
6. Zege EP, Ivanov AP, Katsev IL. Image Transfer Through a Scattering Media. Berlin: Springer-Verlag; 1991.
7. Mishchenko M, Dlugach J, Yanovitskij E, Zakharova N. Bidirectional reflectance of flat, optically thick particulate layers: An efficient radiative transfer solution and applications to snow and soil surfaces. J Quant Spect Rad Trans 1999;63:409–430.
8. Kokhanovsky AA, Zege EP. Scattering optics of snow. Appl Opt 2004;43:1589–1602.
9. Gordon HR. Simple calculation of the diffuse reflectance of ocean. Appl Opt 1973;12:2803, 2804.
10. Hapke B. Theory of Reflectance and Emittance Spectroscopy. New York: Cambridge University Press; 1993.
11. Pierce PE, Marcus RT. Radiative transfer theory solid color-matching calculations. Color Res Appl 1997;22:72–87.
12. van de Hulst HC. The spherical albedo of a planet covered with a homogeneous cloud layer. Astron Astrophys 1974;35:209–214.
13. van de Hulst HC. Multiple Light Scattering, Vol. 2. London: Academic Press; 1980.
14. Ambartsumian VA. On the scattering of light by a diffuse medium. Dokl Akad Nauk SSSR 1943;38:257–265. [Compt Rend (Doklady) Acad Sci USSR 1943;38:229–232]
15. Kokhanovsky AA. Reflection of light from nonabsorbing semi-infinite cloudy media: A simple approximation. J Quant Spectrosc Radiat Transfer 2004;85:25–33.
16. Melnikova IN, Dlugach ZM, Nakajima T, Kawamoto K. Calculation of the reflection function of an optically thick scattering layer for a Henyey–Greenstein phase function. Appl Opt 2000;39:4195–4204.
17. Rozenberg GV. Optical characteristics of thick weakly absorbing scattering layers. Dokl Akad Nauk 1962;145:775–777.
18. Kokhanovsky AA. The accuracy of selected approximations for the reflection function of a semi-infinite turbid medium. J Phys D: Appl Phys 2002;35:1057–1062.
19. Golubitsky VM, Levin IM, Tantashev MV. Brightness coefficient of a semi-infinite layer of sea water. Izvestiya Atmos Ocean Phys 1974; 10:1235–1238.
20. Golubitsky VM, Levin IM. Transmission and reflection of light by a layer with strongly anisotropic scattering. Izvestiya Atmos Ocean Phys 1980;16:1051–1058.
21. Anikonov AS, Ermolaev SY. On diffuse light reflection from a semi-infinite atmosphere with a highly extended phase function. Vestnik LGU 1977;7:132–137.
22. Zege EP. Reflection function of a semi-infinite layer. Izvestiya Ocean Atmos Phys 1983;19:927–936.